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Optimization of the contact pressure in the problem of the interaction of a punch and an elastic medium☆

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ABSTRACT

The problem of optimizing the pressure distribution under a rigid punch, which interacts without friction with an elastic medium filling a half-space, is investigated. The shape of the punch is taken as the initial variable of the design, while the root mean square deviation of the pressure distribution, which occurs under the punch, from a certain specified distribution, plays the role of the minimized functional. The values of the total forces and moments, applied to the punch, are assumed to be given, which leads to limitations imposed on the pressure distribution by the equilibrium conditions. It is shown that the optimization problem allows of decomposition into two successively solvable problems. The first problem consists of finding the pressure distribution which makes the optimized quality functional a minimum. The second problem is reduced to the problem of obtaining directly the optimum shape of the punch that yields the pressure distribution found. The optimization problem is investigated analytically for punches of different shape in plan. The optimum shapes are given in explicit form for punches with rectangular bases.

1. Statement of the optimization problem

The contact interaction of a rigid punch and an elastic medium,[1–6] which occupies a half-space $z \geq 0$, in a rectangular system of coordinates, is considered. The punch is in equilibrium under the action of external forces and the reaction of the elastic medium applied to it. The xy plane, bounding the half-space, contains the contact region Ω_f (the base of the punch) and the region Ω_0, free from forces. The punch surface, in contact with the elastic medium, is described by the equation $z = f(x, y)$, $(x, y) \in \Omega$. Assuming that there are no friction forces in the contact interaction region of the punch and the elastic medium, the boundary conditions of the boundary-value problem of the theory of elasticity, describing the stresses and displacements in the elastic medium, are written in the form[1–5]

$$\begin{aligned} &w = f(x,y), \quad \sigma_{xz} = 0, \quad \sigma_{yz} = 0, \quad (x,y) \in \Omega_f \\ &\sigma_{zz} = 0, \qquad \sigma_{xz} = 0, \quad \sigma_{yz} = 0, \quad (x,y) \in \Omega_0 \end{aligned} \tag{1.1}$$

where w is the component of the displacements vector normal to the xy plane, and σ_{ij} $(i, j = x, y, z)$ are the components of the stress tensor. The contact pressure acting on the punch is described by the function $p = p(x, y)$, i.e.

$$\sigma_{zz} = -p(x,y), \quad (x,y) \in \Omega_f \tag{1.2}$$

For a known pressure distribution $p(x, y)$ $((x, y) \in \Omega_f)$, the resulting force P and moments M_x and M_y about the x and y axes, applied to the punch, are defined by the expressions[2]

$$P(p) = \int p(x,y)\, d\Omega_f, \quad M_y(p) = \int xp(x,y)\, d\Omega_f, \quad M_x(p) = \int yp(x,y)\, d\Omega_f \tag{1.3}$$

Here and henceforth, unless otherwise stated, the integration is carried out over the region Ω_f.

We will assume, as is usually done in contact-interaction mechanics, that the punch shape $f(x, y)$ is continuous and a smooth function of the coordinates. We will therefore only consider those punch shapes which ensure that contact interaction occurs over the whole surface

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of the punch base. For this to occur we require that the following condition is satisfied

$$p = p(x,y) \geq 0, \quad (x,y) \in \Omega_f \tag{1.4}$$

We will introduce into consideration the function $p_g = p_g(x, y) \geq 0$, characterizing the desired pressure distribution under the punch. In certain specific cases this can be a constant pressure or the pressure averaged over the contact area. We will consider the following integral as the optimized functional

$$J = J(p[f]) = \int (p - p_g)^2 d\Omega_f \tag{1.5}$$

where p_g is a specified function. Functional (1.5) represents the mismatch between the pressure distribution $p(x, y)$, corresponding to a certain punch shape $f(x, y)$, and the specified "required" distribution $p_g(x, y)$. In particular, as already mentioned, the value of p_g can be specified to be equal to the average pressure, given by the expression

$$p_m = \frac{1}{S}\int p(x,y)\, d\Omega_f, \quad S = \operatorname{mes}\Omega_f \tag{1.6}$$

where S is the area of the contact region Ω_f.

Assuming the values of the external forces acting or the punch to be specified, we will formulate the following problem of optimizing the punch shape. It is required to determine the function $f(x, y)$ which describes the punch shape and which makes the mismatch functional a minimum, namely

$$J_* = \min_f J(p[f]) = J(p[f_*]) \tag{1.7}$$

when the equilibrium conditions

$$P(p) = P^*, \quad M_y(p) = M_y^*, \quad M_x(p) = M_x^* \tag{1.8}$$

and the contact condition (1.4) are satisfied. Here $P^* \geq 0$, $M_y^* \geq 0$, $M_x^* \geq 0$ are the specified values of the forces and moments applied to the punch.

Optimization problem (1.4), (1.7), (1.8) can be reduced to two problems that are solved successively. The first problem is to find the pressure distribution $p_*(x, y)$ which minimizes the mismatch functional (1.5) for limitations (1.8) and (1.4), imposed on the pressure distribution. The second problem is to determine the optimum punch shape $f_*(x, y)$ for which the pressure distribution $p_*(x, y)$ is obtained.

2. Determination of the optimum pressure

To find the optimum pressure distribution $p_*(x, y)$, which minimizes the functional (1.5), we take into account limitations (1.8), imposed on the integral characteristics, and the local limitation (1.4). To take the local limitation into account, we will represent condition (1.4) in the form

$$p(x,y) - \psi^2(x,y) = 0, \quad (x,y) \in \Omega_f \tag{2.1}$$

where $\psi(x, y)$ is an unknown variable. For the problem of optimizing the pressure formulated above, we construct the extended Lagrange functional

$$J^L = \int (p - p_g)^2 d\Omega_f - \lambda \int p\, d\Omega_f - \alpha \int xp\, d\Omega_f - \beta \int yp\, d\Omega_f - \int \chi\left(p - \psi^2\right) d\Omega_f \tag{2.2}$$

where the constants λ, α and β are Lagrange multipliers and $\chi(x, y)$ is the Lagrange function.[7–10] The conditions for an extremum of functional (2.2) with respect to the variables p and ψ are written as follows:

$$p = p_g + \frac{1}{2}\lambda + \frac{1}{2}\alpha x + \frac{1}{2}\beta y + \frac{1}{2}\chi, \quad \psi\chi = 0, \quad (x,y) \in \Omega_f \tag{2.3}$$

The first condition of (2.3) is the required pressure distribution, in which the unknown quantities λ, α, β and χ occur. Equalities (1.8) serve to determine the constants λ, α and β, while the function χ is found using the second condition of (2.3), on taking which into account we have

$$\text{if } \ \psi(x,y) \neq 0, \text{ then } \chi(x,y) = 0 \ \text{ when } \ (x,y) \in \Omega_f^+ \subset \Omega_f$$

$$\text{if } \ \psi(x,y) = 0, \text{ then } p(x,y) = 0 \ \text{ when } \ (x,y) \in \Omega_f^- \subset \Omega_f$$

These conditions, in general, enable us to split the initial region Ω_f into two regions: Ω_f^+ and Ω_f^- ($\Omega_f^+ + \Omega_f^- = \Omega_f$), where $p(x, y) > 0$ when $(x, y) \in \Omega_f^+$ and $p(x, y) = 0$ when $(x, y) \in \Omega_f^-$. The optimum pressure distribution is given by the expression

$$p_*(x,y) = \max\left\{0, \ p^+(x,y)\right\}; \quad p^+(x,y) = p_g(x,y) + \frac{1}{2}\lambda + \frac{1}{2}\alpha x + \frac{1}{2}\beta y \tag{2.4}$$

The operation max here denotes choosing, for each (x, y), the maximum of the two numbers, written in the braces. Hence, the value of the optimum pressure, defined by equality (2.4), is non-negative.

We will consider the case when the optimum pressure is strictly positive in the contact region, i.e., $\Omega_f = \Omega_f^+$, and is given by the expression

$$p_*(x,y) = p^+(x,y), \quad (x,y) \in \Omega_f \tag{2.5}$$

In this case the unknown Lagrange multipliers λ, α and β are found from the following system of three linear algebraic equations:

$$\begin{aligned} &S\lambda + S_y\alpha + S_x\beta = 2\tilde{P} \equiv 2(P^* - P^g) \\ &S_y\lambda + I_y\alpha + I_{xy}\beta = 2\tilde{M}_y \equiv 2(M_y^* - M_y^g) \\ &S_x\lambda + I_{xy}\alpha + I_x\beta = 2\tilde{M}_x \equiv 2(M_x^* - M_x^g) \end{aligned} \tag{2.6}$$

where we have used the notation

$$\begin{aligned} &S = \int d\Omega_f, \quad S_x = \int y d\Omega_f, \quad S_y = \int x d\Omega_f \\ &I_x = \int y^2 d\Omega_f, \quad I_y = \int x^2 d\Omega_f, \quad \mathrm{I}_{xy} = \int xy d\Omega_f \\ &P^g = \int p_g d\Omega_f, \quad \mathrm{M}_y^g = \int x p_g d\Omega_f, \quad \mathrm{M}_x^g = \int y p_g d\Omega_f \end{aligned} \tag{2.7}$$

If the contact region Ω_f possesses symmetry about the x and y axes, then, as can easily be shown, using equalities (2.6) and (2.7),

$$S_x = 0, \quad S_y = 0, \quad I_{xy} = 0 \tag{2.8}$$

while for the Lagrange multipliers the following expressions hold

$$\lambda = 2\tilde{P}/S, \quad \alpha = 2\tilde{M}_y/I_y, \quad \beta = 2\tilde{M}_x/I_x \tag{2.9}$$

If moreover the function $p_g(x, y)$ is symmetrical about the x and y axes, i.e.,

$$p_g(x,y) = p_g(-x,y), \quad p_g(x,y) = p_g(x,-y), \quad (x,y) \in \Omega_f \tag{2.10}$$

the following inequalities will be satisfied

$$M_y^g = 0, \quad M_x^g = 0, \quad \alpha = 2M_y^*/I_y, \quad \beta = 2M_x^*/I_x \tag{2.11}$$

In the case when the average pressure is taken as the specified pressure distribution, the following relations are satisfied

$$p_g = p_m = \frac{P^*}{S}, \quad \mathrm{P}^g = P^*, \quad \tilde{P} = 0 \tag{2.12}$$

and, consequently, we will have

$$\lambda = 0, \quad p_*(x,y) = \frac{P^*}{S} + \frac{1}{2}\alpha x + \frac{1}{2}\beta y \tag{2.13}$$

Before we construct the optimum punch shapes, corresponding to the optimum pressure distribution obtained, we point out that the case of complete contact being considered ($p_*(x, y) > 0$ in the region Ω_f) is possible under certain limitations on the problem parameters.

We will denote by (x^0, y^0) the point of the region Ω_f, for which the expression for $p^+(x, y)$ reaches a minimum, i.e.,

$$(x^0, y^0) = \underset{(x,y)\in\Omega_f}{\operatorname{argmin}} \left(p_g(x,y) + \frac{1}{2}\lambda + \frac{1}{2}\alpha x + \frac{1}{2}\beta y \right)$$

Then, when the region Ω_f and the function $p_g(x, y)$ are symmetrical about the x and y axes, we will have the following condition, which ensures contact in the region Ω_f

$$P^* > \int_{\Omega_f} p_g(x,y) d\Omega_f - S\left(p_g\left(x^0, y^0\right) + \frac{M_y^*}{I_y} x^0 + \frac{M_x^*}{I_x} y^0 \right)$$

In the case of a rectangular region Ω_f: $-a \le x \le a, \ -b \le y \le b$ and $p_g = P^*/S$, to obtain complete contact, as follows from relations (2.5), (2.11)–(2.13), it is necessary to satisfy the inequality

$$P^* \ge 3M_x^*/b + 3M_y^*/a$$

Note also that the construction of the optimum distribution $p_*(x, y)$ is based on the necessary conditions for an extremum. A check of the sufficient conditions for a minimum of the functional being optimized requires additional estimates, which are not, in the general case, given here. For brevity, we will only consider a special case, when the moments applied to the punch are equal to zero ($M_x^* = M_y^* = 0$) and $p_g = P^*/S$. In this case, by relations (2.5) and (2.9)–(2.12) $J(p_*) = 0$. Taking into account the fact that $J(p) \ge 0$ for all the admissible forms of the pressure, we arrive at the conclusion that $p_* = P^*/S$ gives a minimum of the functional being optimized in the case considered.

3. Determination of the optimum punch shape

As is well known (see, for example, Refs 4–6), for a pressure distribution $p(x, y)$ specified in the region Ω_f, the displacements of the points of the elastic half-space $w(x, y, z)$ in the direction of the z axis are found using the potential of a simple layer $\omega(x, y, z)$ and are written in the form

$$w(x,y,z) = \kappa\omega(x,y,z) = \kappa\int\frac{p(x',y')}{\sqrt{(x-x')^2+(y-y')^2+z^2}}d\Omega'_f, \quad \kappa = \frac{1-\nu^2}{\pi E} \tag{3.1}$$

where the primes denote quantities over which the integration is performed. Consequently, the optimum punch shape, corresponding to the optimum pressure distribution $p_*(x', y')$, $((x', y') \in \Omega_f)$, can be represented in the form $(d\Omega'_f = dx'\,dy')$

$$f_*(x,y) = \kappa[\omega(x,y,0)]_{p=p_*} = \kappa\int\frac{p_*(x',y')}{\rho(x,y,x',y')}d\Omega'_f$$
$$\rho(x,y,x',y') = \sqrt{(x-x')^2+(y-y')^2} \tag{3.2}$$

We will introduce the polar coordinates ρ, θ with centre at the observation point $N(x, y)$, $(x, y) \in \Omega_f$ (Fig. 1). Here the coordinates of the fixed point $N(x, y)$ and of the point (x', y'), which changes position in the region Ω_f during integration, will be connected by the relations

$$x' - x = \rho\cos\theta, \quad y' - y = \rho\sin\theta \tag{3.3}$$

Introducing further the notation

$$p_*(x',y') = C + Ax' + By' = C + Ax + By + \rho(A\cos\theta + B\sin\theta)$$
$$C = p_g + \frac{1}{2}\lambda, \quad A = \frac{1}{2}\alpha, \quad \mathrm{B} = \frac{1}{2}\beta \tag{3.4}$$

and changing to polar coordinates (3.3) when integrating in equality (3.2), we will have

$$f_*(x,y) = \kappa\int_0^{2\pi}d\theta\int_0^{R(\theta)}p_*(x+\rho\cos\theta,\ y+\rho\sin\theta)d\rho$$
$$= \kappa(C + Ax + By)D_0(x,y) + \frac{1}{2}\kappa AD_c(x,y) + \frac{1}{2}\kappa BD_s(x,y) \tag{3.5}$$

Here

$$D_0(x,y) = \int_0^{2\pi}Rd\theta, \quad D_c(x,y) = \int_0^{2\pi}R^2\cos\theta d\theta, \quad D_s(x,y) = \int_0^{2\pi}R^2\sin\theta d\theta \tag{3.6}$$

Note that when using expression (3.2) and carrying out the integration in equality (3.5) we have taken into account the relation $(d\Omega'_f = \rho d\rho d\theta)$ for the element of the region Ω_f considered. Hence, the problem of finding the optimum shape of the punch, which interacts without friction with an elastic base, has been reduced to finding the geometrical characteristics D_0, D_c and D_s, that depend on the shape of the contact region Ω_f.

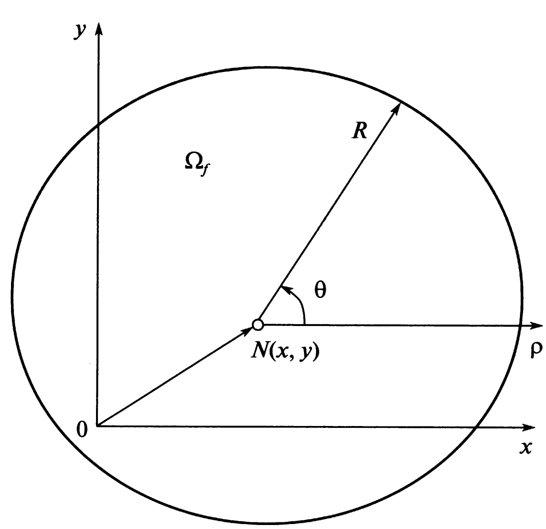

Fig. 1.

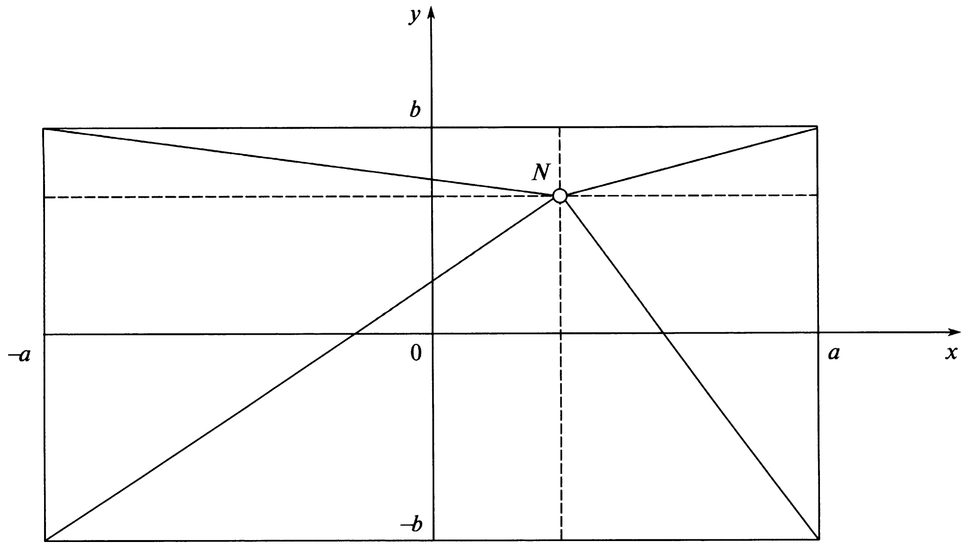

Fig. 2.

As an example we will consider the optimization problem for a punch having a rectangular shape in plan with sides $2a$ and $2b$. By superposing the origin of a Cartesian system of coordinates with the centre of the rectangle (Fig. 2), we will have $\Omega_f = \{-a \le x \le a, \ -b \le y \le b\}$, where $a > 0$ and $b > 0$ are specified parameters. Combining the point $N(x, y)$ with the vertices of the rectangle and dropping perpendiculars from the point $N(x, y)$ (shown by the dashed lines in Fig. 2) onto the sides of the rectangle, we divide the rectangle considered into eight right-angled triangles with heights h_i ($i = 1, 2, \ldots, 8$) at the vertex $N(x, y)$.

The calculation of the integrals D_0, D_c and D_s on the boundary of the region Ω_f reduces to calculating them along the sides of the triangles lying on the boundary of the region Ω_f. Here the boundaries of the triangles are given by the expressions $R(\theta) = h_i/\cos\theta$, where h_i is the height of the corresponding triangle, while the required integrals D_0, D_c and D_s are represented in the form of the following sums

$$D_0 = \sum_{i=1}^{8} \Delta_i^0(\gamma_i), \quad D_c = \sum_{i=1}^{8} \Delta_i^c(\gamma_i), \quad D_s = \sum_{i=1}^{8} \Delta_i^s(\gamma_i) \tag{3.7}$$

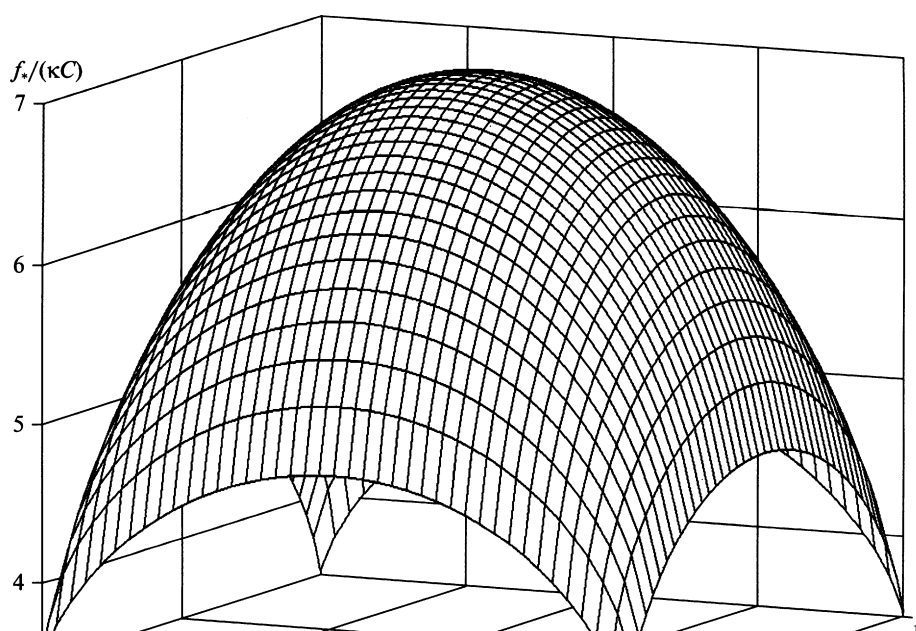

Fig. 3.

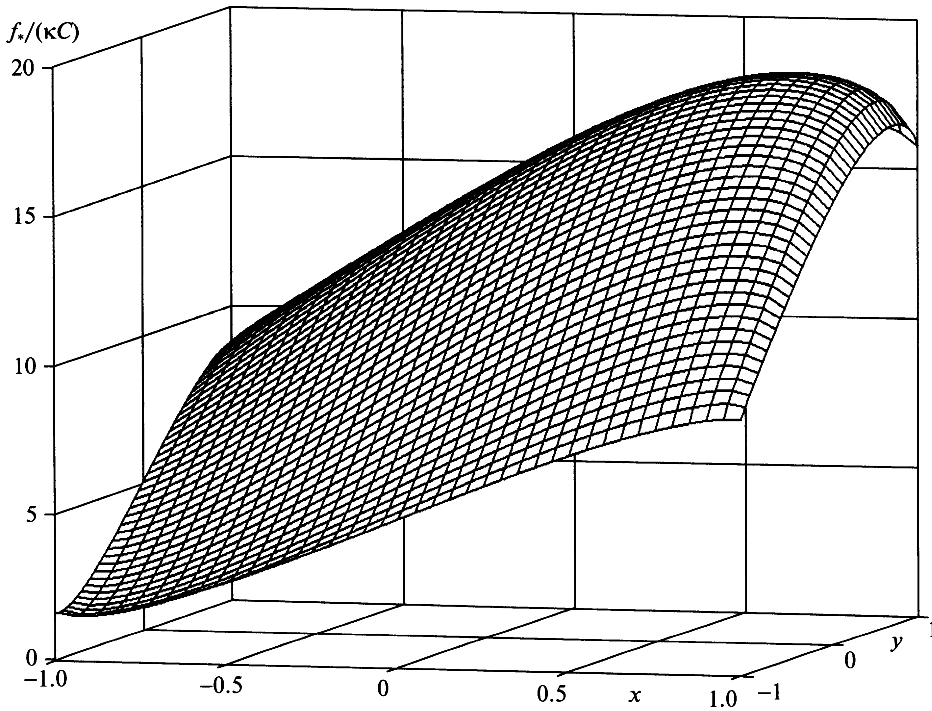

Fig. 4.

where

$$\begin{aligned}
\Delta_i^0(\gamma_i) &= \int_0^{\gamma_i} R d\theta = h_i \int_0^{\gamma_i} \frac{d\theta}{\cos\theta} = \frac{1}{2} h_i \ln \frac{1+\sin\gamma_i}{1-\sin\gamma_i} \\
\Delta_i^c(\gamma_i) &= \int_0^{\gamma_i} R^2 \cos\theta d\theta = h_i^2 \int_0^{\gamma_i} \frac{d\theta}{\cos\theta} = h_i^2 \ln \frac{1+\sin\gamma_i}{1-\sin\gamma_i} \\
\Delta_i^s(\gamma_i) &= \int_0^{\gamma_i} R^2 \sin\theta d\theta = h_i^2 \int_0^{\gamma_i} \frac{\sin\theta d\theta}{\cos^2\theta} = h_i^2 \left(\frac{1}{\sqrt{1-\sin^2\gamma_i}} - 1 \right)
\end{aligned} \tag{3.8}$$

The heights of the triangles considered (Fig. 2) and the angles at their vertices at the point $N(x, y)$ are given by the expressions

$$\begin{aligned}
& h_1 = h_2 = a - x, \quad h_3 = h_4 = b - y, \quad h_5 = h_6 = a + x, \quad h_7 = h_8 = b + y \\
& \sin\gamma_1 = \frac{b+y}{\sqrt{(a-x)^2 + (b+y)^2}}, \quad \ldots
\end{aligned} \tag{3.9}$$

The sines of the angles $\gamma_2, \gamma_3, \ldots, \gamma_8$ are defined similarly. Using expressions (3.7) and (3.9), we have, after elementary reduction,

$$\begin{aligned}
D_0 &= \frac{1}{2}[(a-x)\varphi(b,a;y,-x) + (b-y)\varphi(a,b;-x,-y) \\
&+ (a+x)\varphi(b,a;-y,x) + (b+y)\varphi(a,b;x,y)] \\
D_c &= (a-x)^2\varphi(b,a;y,-x) + (b-y)^2\varphi(a,b;-x,-y) \\
&+ (a+x)^2\varphi(b,a;-y,x) + (b+y)^2\varphi(a,b;x,y) \\
D_s &= g(b,a;y,-x) + g(a,b;-x,-y) + g(b,a;-y,x) + g(a,b;x,y)
\end{aligned} \tag{3.10}$$

where

$$\varphi(a,b;x,y) = \ln\left[f(a,b;x,y)\,f(a,b;-x,y)\right]$$

$$f(a,b;x,y) = \frac{\sqrt{(a+x)^2+(b+y)^2}+a+x}{\sqrt{(a+x)^2+(b+y)^2}-a-x}$$

$$g(a,b;x,y) = (b+y)\left[\sqrt{(a+x)^2+(b+y)^2}+\sqrt{(a-x)^2+(b+y)^2}-2b-2y\right]$$

The optimum punch shape $F = f_*/(\kappa C) = D_0(x, y)$, shown in Fig. 3, in the case when the external forces are symmetrical ($P^* \neq 0$, $M_y^* = M_x^* = 0$), corresponds to values of the problem parameters $a = b = 1$ and $A = B = 0$. An example of an asymmetrical optimum shape of the punch $F = f_*/(\kappa C)$, shown in Fig. 4 in the case when $P^* \neq 0$, $M_y^* \neq 0$, $M_x^* \neq 0$, corresponds to values of the parameters $a = b = 1$ and $A = B = C = 1$.

The approach used in this problem, based on decomposition of the initial optimization problem, can also be used when considering the more general contact problem, when the external loads are applied not only directly to the punch but also act on parts of the surface of the elastic medium situated outside the contact region.

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